Chapter 4 Waves in Unbounded Medium

4.1 Electromagnetic Sources

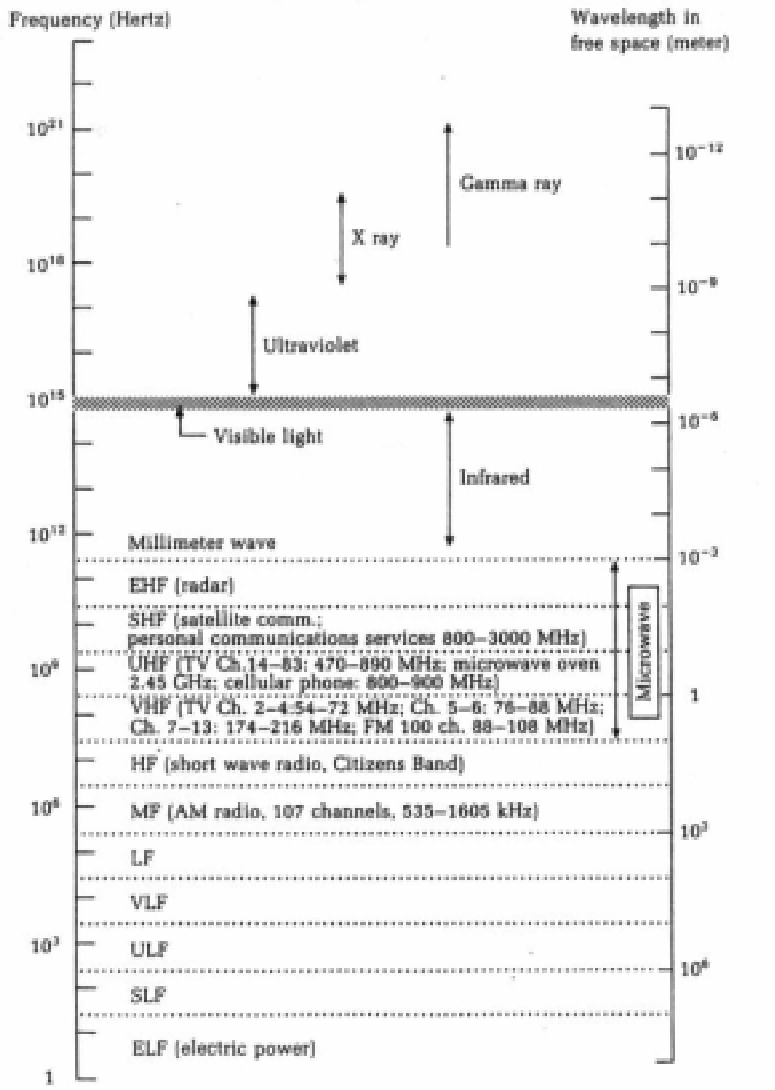


Figure 3.1 Electromagnetic spectrum.

4.2 Uniform plane waves in free space

Maxwell’s equation in free space is given by:

$$rot\mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \quad (4.2.1)$$

$$rot\mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (4.2.2)$$

$$div\mathbf{E} = 0 \quad (4.2.3)$$

$$div\mathbf{H} = 0 \quad (4.2.4)$$

which is satisfied by electromagnetic filed in source free space. In this notation, the constitutive relations are used.

$$\mathbf{D} = \varepsilon_0 \mathbf{E} \quad (4.2.5)$$

$$\mathbf{B} = \mu_0 \mathbf{H} \qquad (4.2.6)$$

(4.2.1)-(4.2.4) can be considered as simultaneous differential equations having two unknowns, i.e., $\boldsymbol{E}$ and $\boldsymbol{H}$. We will delete one of the unknowns from the four equations. Since $\boldsymbol{E}$ and $\boldsymbol{H}$ have the same form, it can be done in the same manner. Here, we will deleted $\boldsymbol{H}$ first.

$$rot(rot\mathbf{E}) = -rot(\mu_0 \frac{\partial \mathbf{H}}{\partial t})$$
$$= -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad (4.2.6)$$

And using the vector identity,

$$rot(rot\mathbf{E}) = grad(\underline{\underline{div\mathbf{E}}}) - \Delta\mathbf{E} \qquad (div\mathbf{E} = 0)$$
$$= -\Delta\mathbf{E} \qquad (4.2.7)$$

where Δ indicates Laplacian to vector, and we have

$$\Delta \boldsymbol{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0 \qquad (4.2.8)$$
$$\Delta \boldsymbol{H} - \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{H}}{\partial t^2} = 0 \qquad (4.2.9)$$

The time-harmonic representation of the wave equation (4.2.8) and (4.2.9) are give as:

$$\Delta \boldsymbol{E} - \omega^2 \mu_0 \varepsilon_0 \boldsymbol{E} = 0 \qquad (4.2.10)$$
$$\Delta \boldsymbol{H} - \omega^2 \mu_0 \varepsilon_0 \boldsymbol{H} = 0 \qquad (4.2.11)$$

In the Cartesian coordinate, the Laplacian can be defined as:

$$\Delta = \mathbf{i}_x \frac{\partial^2}{\partial x^2} + \mathbf{i}_y \frac{\partial^2}{\partial y^2} + \mathbf{i}_z \frac{\partial^2}{\partial z^2} \qquad (4.2.12)$$

and the vector notation in (4.2.8)(4.2.9) can be re-written as:

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} - \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} = 0 \qquad (4.2.13)$$
$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} - \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} = 0 \qquad (4.2.14)$$
$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} - \mu_0 \varepsilon_0 \frac{\partial^2 E_z}{\partial t^2} = 0 \qquad (4.2.15)$$

It should be noticed that (4.2.13)-(4.2.15) are second-order differential equations of electrical field in time and space. This form of equation is generally refereed as wave equations. The time-varying electromagnetic field in free space have to satisfy the wave equation of (4.2.13)-(4.2.15).

4.3 Plane-wave solution

Assume the followings and solve (4.2.13)-(4.2.15).

Assumption (1) : Wave propagates to z direction.

(2) : The wave is uniform in the x-y plane. This is mathematically identical to : $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$

By using the assumption (2), (4.2.13)-(4.2.15) can be simplified as:

$$\frac{\partial^2 E_x}{\partial z^2} - \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} = 0 \qquad (4.3.1)$$

$$\frac{\partial^2 E_y}{\partial z^2} - \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} = 0 \qquad (4.3.2)$$

$$\frac{\partial^2 E_z}{\partial z^2} - \mu_0 \varepsilon_0 \frac{\partial^2 E_z}{\partial t^2} = 0 \qquad (4.3.3)$$

Without loosing the generality, we can rotate the coordinate and have:

$$E_y = 0, E_x \neq 0 \qquad (4.3.4)$$

Then we have non-zero solution of E_x and E_z. Now we come back to the Maxwell's equation.

$$rot\boldsymbol{E} = \begin{vmatrix} \boldsymbol{i}_x & \boldsymbol{i}_y & \boldsymbol{i}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & 0 & E_z \end{vmatrix} = \boldsymbol{i}_y \frac{\partial E_x}{\partial z}$$

$$= -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} = -\boldsymbol{i}_y \mu_0 \frac{\partial H_y}{\partial t} \qquad (4.3.5)$$

This equation indicates that $H_y \neq 0, H_x = H_z = 0$, and

$$rot\boldsymbol{H} = \begin{vmatrix} \boldsymbol{i}_x & \boldsymbol{i}_y & \boldsymbol{i}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & H_y & 0 \end{vmatrix} = -\boldsymbol{i}_x \frac{\partial H_y}{\partial z}$$

$$= \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{i}_x \varepsilon_0 \frac{\partial E_x}{\partial t} \qquad (4.3.6)$$

shows that $E_x \neq 0, E_z = 0$. Then we can find that only

$$\frac{\partial^2 E_x}{\partial z^2} - \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} = 0 \qquad (4.3.7)$$

gives the non-zero plane wave solution.

4.4 General solution to the plane-wave equation

The general solution to (4.3.7) is given by

$$E_x = f_1(z - ct) + f_2(z + ct) \qquad (4.4.1)$$

where

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 (m/s) \qquad (4.4.2)$$

is the velocity of light in free space, and f_1, f_2 are arbitrary functions. f_1, f_2 indicates that the wave solution propagate without deforming at the velocity of c. By substituting (4.4.1) into $rot\mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$ we obtain

$$H_y = \frac{1}{\eta_0}\left[f_1(z-ct) - f_2(z+ct) \right] \quad (4.4.3)$$

where

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cong 376.6 \cong 120\pi \quad (4.4.4)$$

is the intrinsic impedance of free space.

The time-harmonic wave equation can be obtained from (4.3.7) as:

$$\frac{\partial^2 E_x}{\partial z^2} + \mu_0 \varepsilon_0 \omega^2 E_x = \frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0 \quad (4.4.5)$$

$$k = \omega\sqrt{\mu_0 \varepsilon_0} = \frac{\omega}{c} \qquad \text{wave number} \quad (4.4.6)$$

and the general solution to (4.4.6) is given by

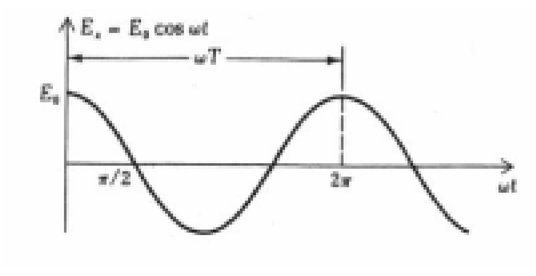
$$E_x = A_1 e^{-jkz} + A_2 e^{+jkz} \quad (4.4.7)$$

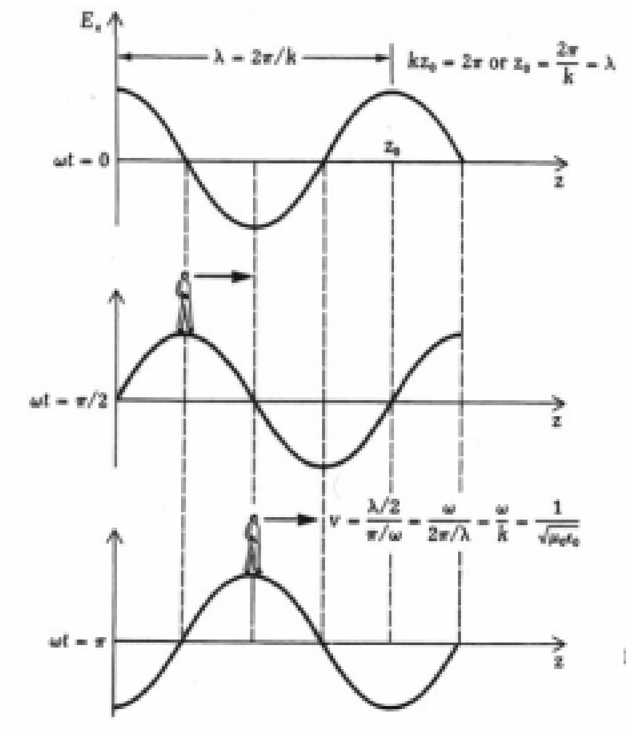
(4.4.6) is called dispersion relation, which determines the wave number of the solution of the wave equation. By re-writing this time-harmonic notation in time-domain, we have

$$E_x(z,t) = \mathrm{Re}\left\{\left(A_1 e^{-jkz} + A_2 e^{+jkz}\right) e^{j\omega t}\right\} = A_1 \cos(\omega t - kz) + A_2 \cos(\omega t + kz) \quad (4.4.8)$$

and this equation indicates a sinusoidal wave propagating to the z-direction at the velocity of *c*. The wavelength λ satisfies

$$\lambda = \frac{2\pi}{k} \quad (4.4.9)$$





The figures show how the solution, which can now be recognized as a sinusoidal wave, propagates with time. Imagine we ride along with the wave. At what velocity shall we move in order to keep up with the wave? Mathematically, the phase argument of the term $\cos(\omega t - kz)$ must be constant. That is,

$$\omega t - kz = \text{a constant} \quad (4.4.10)$$

So the velocity of propagation is given by

$$\frac{dz}{dt} = v = \frac{\omega}{k} \qquad (4.4.11)$$

From (4.4.6) we obtain the phase velocity

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \qquad (4.4.12)$$

The plane wave solution, which propagates to z-direction, then can be written as;

$$\boldsymbol{E} = \hat{\boldsymbol{x}} E_0 e^{-jkz} \qquad (4.4.13)$$

$$\boldsymbol{H} = \hat{\boldsymbol{y}} \sqrt{\frac{\varepsilon_0}{\mu_0}} E_0 e^{-jkz} \qquad (4.4.14)$$

And the Poyinting vector is

$$\boldsymbol{S} = \hat{\boldsymbol{z}} \frac{E_0^2}{\eta_0} \cos^2(\omega t - kz) \qquad (4.4.15)$$

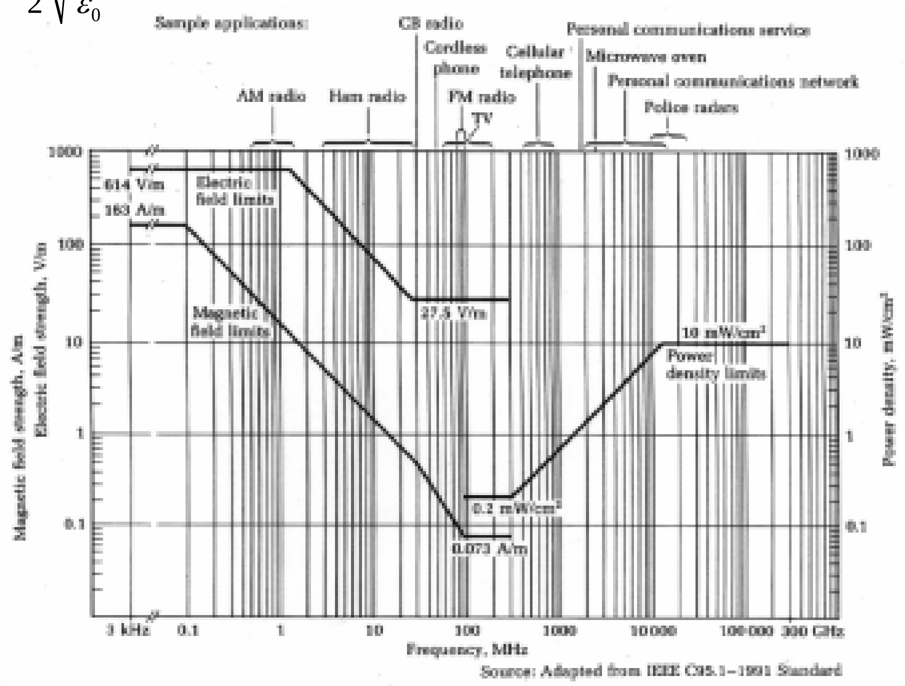
Notice that in all the above discussions, the solution in (4.4.7),(4.4.8) or equivalently (4.4.13) are independent of a and *y* coordinates. In other words for an observer anywhere in the *x*-*y* plane with the same value for *z*, the phenomena are the same. The constant phase front is defined by setting the phase in (4.4.7) equal to a constant. We have

$$kz = C \qquad (4.4.16)$$

where *C* is a constant defining a plane perpendicular to the *z* axis at $z = C/k$. We call waves whose phase fronts are planes plane waves. A plane wave with uniform amplitudes over its constant-phase planes is called a uniform plane wave.

The power density carried by a uniform wave is calculated by the time-average Poynting vector.

$$\langle \boldsymbol{S} \rangle = \frac{1}{2} \mathrm{Re}\{\boldsymbol{E} \times \boldsymbol{H}^*\} = \hat{\boldsymbol{z}} \frac{1}{2} \sqrt{\frac{\mu_0}{\varepsilon_0}} |E_0|^2 \qquad (4.4.17)$$



4.5 Polarization

The uniform plane wave discussed in the previous section is

$$\boldsymbol{E}(z,t) = E_0 \cos(\omega t - kz)\hat{\boldsymbol{x}} \qquad (4.5.1)$$

Tracing the tip of the vector $\boldsymbol{E}$ at any point *z* will show that the tip always stays on the *x* axis with maximum displacement E_0. We thus conclude that the uniform plane wave is linearly polarized.

Now we consider a plane wave with the following electric-filed vector.

$$\boldsymbol{E} = \hat{x} a e^{-j(kz-\phi_a)} + \hat{y} b e^{-j(kz-\phi_b)} \qquad (4.5.2)$$

The real time-space $\boldsymbol{E}$ vector in (4.5.2) has *x* and *y* components:

$$E_x = a\cos(\omega t - kz + \phi_a) \qquad (4.5.3)$$

$$E_y = b\cos(\omega t - kz + \phi_b) \qquad (4.5.4)$$

where *a* and *b* are real constants. To determine the locus of the tip of $\boldsymbol{E}$ vector in the *x-y* plane as a function of time at any *z*, we can eliminate the variable $(\omega t - kz)$ from (4.5.3)(4.5.4) to obtain an equation for E_x and E_y.

(1)Liner Polarization

$$\phi = \phi_a - \phi_b = 0, \pi \qquad (4.5.5)$$

$$E_y = \pm(\frac{b}{a})E_x \qquad (4.5.6)$$

(2)Circular Polarization

$$\phi = \phi_a - \phi_b = \pm\frac{\pi}{2} \qquad (4.5.7)$$

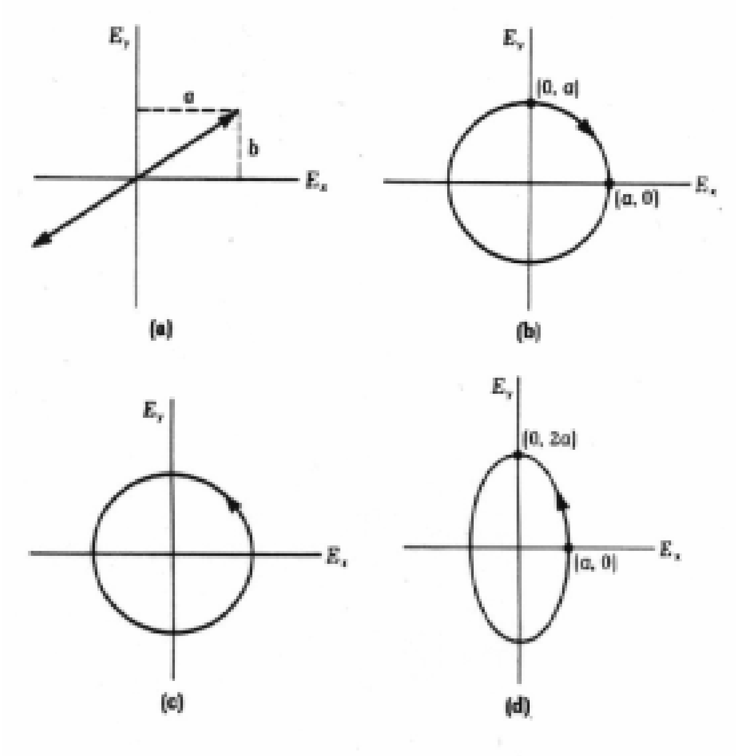
$$A = \frac{b}{a} = 1 \qquad (4.5.8)$$

and

$$E_x = a\cos(\omega t - kz + \phi_a) \qquad (4.5.9)$$

$$E_y = -b\sin(\omega t - kz + \phi_a) \qquad (4.5.10)$$

$$E_x{}^2 + E_y{}^2 = a^2 \qquad (4.5.11)$$



(3)Elliptical Polarization

The wave (4.5.2) is elliptically polarized when it is neither linearly nor circularly polarized. Consider the case $\phi = -\pi/2$ and $A = b/a = 2$.

$$E_x = a\cos(\omega t - kz + \phi_a) \qquad (4.5.12)$$
$$E_y = 2a\sin(\omega t - kz + \phi_a) \qquad (4.5.13)$$

By eliminating *t* yields

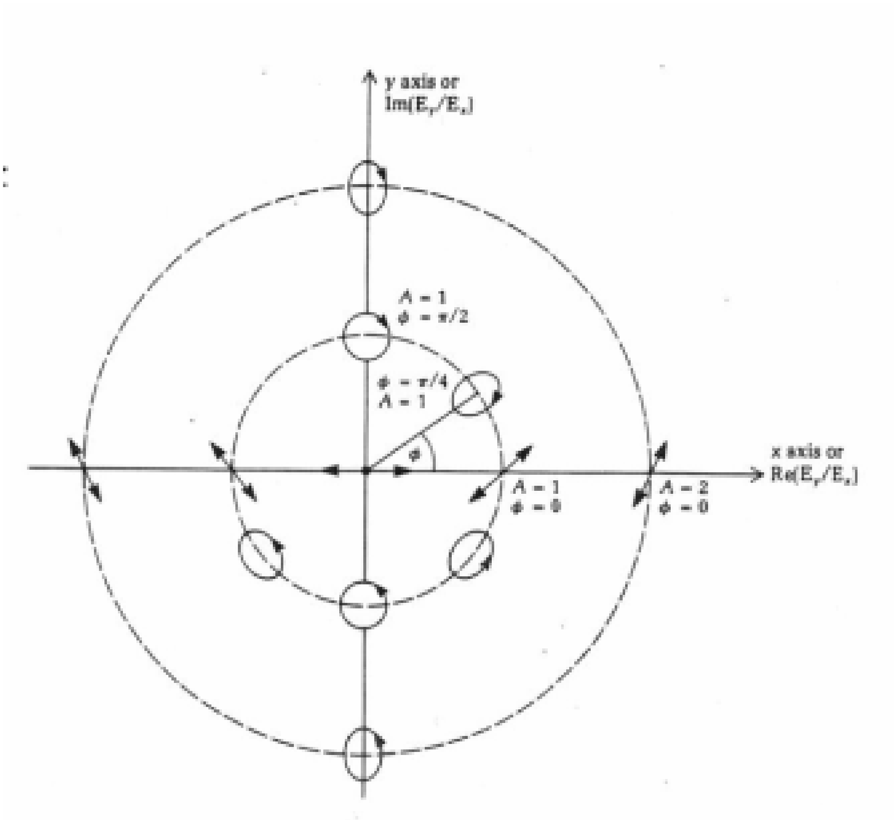
$$\left(\frac{E_x}{a}\right)^2 + \left(\frac{E_y}{2a}\right)^2 = 1 \qquad (4.5.14)$$

Let the complex electric filed be

$$\boldsymbol{E} = (\hat{\boldsymbol{x}}E_x + \hat{\boldsymbol{y}}E_y)e^{-jkz} \qquad (4.5.15)$$
$$\frac{E_y}{E_x} = Ae^{j\phi} \qquad (4.5.16)$$

then, all the polarization status can be expressed by using (A, ϕ).

The figure shows the polarization chart in x-y plane or complex $\frac{E_y}{E_x} = Ae^{j\phi}$ plane.



4.6 Plane wave in dissipative media

The most material has electrical conductivity, and its effect is characterized by the conductivity σ. Ohm's law states that

$$\boldsymbol{J}_c = \sigma \boldsymbol{E} \qquad (4.6.1)$$

where $\boldsymbol{J}_c$ denotes the conducting current. From Ampere's law,

$$rot\boldsymbol{H} = \boldsymbol{J} + j\omega \boldsymbol{D} \qquad (4.6.2)$$

we can see that current density $\boldsymbol{J}$ can embody two kinds of currents- namely, the source current $\boldsymbol{J}_0$ and the conducting current $\boldsymbol{J}_c$. Then, in the conducting medium, the Ampere's law becomes

$$rot\boldsymbol{H} = j\omega(\varepsilon - j\frac{\sigma}{\omega})\boldsymbol{E} + \boldsymbol{J}_0 \qquad (4.6.3)$$

Thus we define

$$\varepsilon = \varepsilon - j\frac{\sigma}{\omega} \qquad (4.6.4)$$

the conductivity becomes the imaginary part of the complex permittivity ε. Then the Maxwell's equation in a conducting medium devoid of any source can be written as

$$rot\boldsymbol{E} = -j\omega\mu \boldsymbol{H} \qquad (4.6.5)$$

$$rot\boldsymbol{H} = j\omega\varepsilon \boldsymbol{E} \qquad (4.6.6)$$

$$div\boldsymbol{H} = 0 \qquad (4.6.7)$$

$$div\boldsymbol{E} = 0 \qquad (4.6.8)$$

where ε is the complex permittivity. The wave equation derived from (4.6.5)-(4.6.8) can be written as

$$\Delta \boldsymbol{E} - \omega^2 \mu\varepsilon \boldsymbol{E} = 0 \qquad (4.6.9)$$

The set of plane-wave fields is still a solution of Maxwell's equations. Namely,

$$\boldsymbol{E} = \hat{\boldsymbol{x}} E_0 e^{-jkz} \qquad (4.6.10)$$

$$\boldsymbol{H} = \hat{\boldsymbol{y}} \sqrt{\frac{\varepsilon}{\mu}} E_0 e^{-jkz} \qquad (4.6.11)$$

where

$$\boldsymbol{k}^2 = \omega^2 \mu\varepsilon \qquad (4.6.12)$$

is the dispersion relation derived from the wave equation and

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \qquad (4.6.13)$$

is the intrinsic impedance of the isotropic media. Note that $\boldsymbol{k}$ and η are now complex numbers.

Because $\boldsymbol{k}$ and η are now complex numbers, we can rewrite these parameters as:

$$\boldsymbol{k} = k_R - jk_I \tag{4.6.14}$$

$$\boldsymbol{\eta} = |\boldsymbol{\eta}| e^{j\phi} \tag{4.6.15}$$

Substitution of (4.6.14) in (4.6.10) and (4.6.11) yields

$$\boldsymbol{E} = \hat{\boldsymbol{x}} E_0 e^{-k_I z} e^{-jk_R z} = \hat{\boldsymbol{x}} E_x \tag{4.6.16}$$

$$\boldsymbol{H} = \hat{\boldsymbol{y}} \sqrt{\frac{\mu}{\varepsilon}} E_0 e^{-k_I z} e^{-jk_R z} e^{-j\phi} \tag{4.6.17}$$

The instantaneous value is $E_x(z,t) = \mathrm{Re}(E_x e^{j\omega t})$ and from (4.6.16) we find

$$\boldsymbol{E_x} = \boldsymbol{E}_0 \boldsymbol{e}^{-\boldsymbol{k_I z}} \cos(\boldsymbol{\omega t} - \boldsymbol{k_R z}) \tag{4.6.18}$$

The above equation represents a wave traveling in the *z* direction with a velocity equal to *v*, where

$$v = \frac{\omega}{k_R} \tag{4.6.19}$$

As the wave travels, the amplitude is attenuated exponentially at the rate k_I nepers per meter.

We define a penetration depth d_p such that, when $k_I z = k_I d_p = 1$, the amplitude of the electric field shown in (4.6.18) will decay to *1/e* of its value at *z=0.*

$$d_p = \frac{1}{k_I} \tag{4.6.20}$$

For conducting media, we have

$$k = \omega\sqrt{\mu\varepsilon}\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]^{\frac{1}{2}} = k_R - jk_I \tag{4.6.21}$$

$$\begin{pmatrix} k_R \\ k_I \end{pmatrix} = \left[\frac{\omega^2\mu\varepsilon}{2}\left\{\left(1 + \frac{\sigma^2}{\omega^2\varepsilon^2}\right) \pm 1\right\}\right]^{\frac{1}{2}} \tag{4.6.22}$$

where $\sigma / \omega\varepsilon$ is called the loss tangent of the conducting media.

(1) Waves in Non-dissipative Media

In material, where the electrical conductivity is zero, the wave behaves similar to that in free space. However, due to μ and ε, the speed of the wave is given by

$$v = 1/\sqrt{\varepsilon\mu} \tag{4.6.23}$$

(2) Waves in Slightly Conducting Media

A slightly conducting medium is one for which $\sigma / \omega\varepsilon << 1$. The value of k in (4.6.21) can be approximated by

$$k = \omega\sqrt{\mu\varepsilon}\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]^{\frac{1}{2}} = \omega\sqrt{\mu\varepsilon}(1 - j\frac{\sigma}{2\omega\varepsilon}) \qquad (4.6.24)$$

and we find

$$k_R = \omega\sqrt{\mu\varepsilon} \qquad (4.6.25)$$

$$k_I = \frac{\sigma}{2}\sqrt{\frac{\varepsilon}{\mu}} \qquad (4.6.26)$$

In this condition, the penetration depth is given by

$$d_p = \frac{2}{\sigma}\sqrt{\frac{\mu}{\varepsilon}} \qquad (4.6.27)$$

(3) Waves in Highly Conducting Media

A highly conducting medium is also called a good conductor, for which $\sigma / \omega\varepsilon >> 1$. In this case, the wave number *k* can be approximated by

$$k = \omega\sqrt{\mu\varepsilon}\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]^{\frac{1}{2}} = \omega\sqrt{\mu\varepsilon\left(\frac{\sigma}{2}\right)}(1 - j) \qquad (4.6.28)$$

Therefore, the penetration depth becomes

$$d_p = \sqrt{\frac{2}{\omega\mu\sigma}} \equiv \delta \qquad (4.6.29)$$

The symbol δ signifies that d_p is so small it is better called the skin depth δ. For a good conductor, the conducting current concentrates on the surface and very little flows inside the conductor. This phenomena is called the skin effect.

In the above discussion, we used the complex permittivbity defined by (4.6.4). However, if we substitute (4.6.4) into (4.6.9), we have

$$\Delta\boldsymbol{E} - \omega^2\mu\varepsilon\boldsymbol{E} - j\omega\mu\sigma\boldsymbol{E} = 0 \qquad (4.6.30$$

If we neglect the third term in (4.6.30) his is a wave equation given in (2.1.1) and if we neglect the second term, this is a diffusion equation given in (2.3.8). The loss tangent $\sigma / \omega\varepsilon$ defines the contribution of the second and the third terms:. It means that, the in Nondispersive media, the electromagnetic wave is governed by the wave equation, and in highly conducting media, it is governed by a diffusion equation. The time harmonic solution of wave equation gives the unified solution to the wave equation, when in highly conducting media, but the physical properties of electromagnetic wave is quite different.