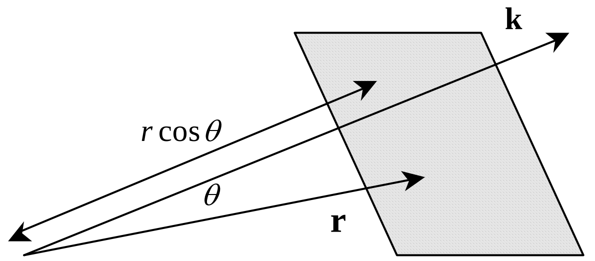
5.1 Plane wave representation in three dimensional apace

Now we consider a plane wave propagating in a direction other than *z*. Now we expand the expression of (4.4.8) into a general form. Taking a component which propagate to one direction in (4.4.8) and write:

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{e}^{(1)} E_0 \cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t) \qquad (5.1.1)$$

where $\boldsymbol{e}^{(1)}$ is a unit vector of electrical field $\boldsymbol{E}$ and $\boldsymbol{k}$ is a wave, which is now represented in three dimensional



space. $\mathbf{r}$ is a position vector of observation point. $(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$ is a scalar value, which defines the phase of the wave. When the phase takes a constant value α,

$$\mathbf{k}\cdot\mathbf{r} = \alpha + \omega t \qquad (5.1.2)$$

is satisfied. when we fix a time as $t = t_1$,

$$\mathbf{k}\cdot\mathbf{r} = \alpha + \omega t_1 \qquad (5.1.3)$$

$\mathbf{r}$ which satisfies the above condition forms a plane which is perpendicular to the vector $\boldsymbol{k}$. If $t = t_1 + \Delta t$, then

$$\boldsymbol{k}\cdot\boldsymbol{r} = \alpha + \omega(t_1 + \Delta t) \qquad (5.1.4)$$

forms a plane, which is parallel to the plane defined by (5.1.3), but at different position. The plane which is determined by the phase $(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$ propagates to a direction of $\boldsymbol{k}$, when time marches. This plane is called a constant phase front. (5.1.1) represents a plane wave which has the electric field to the direction of $\boldsymbol{e}^{(1)}$ and propagates to the direction of the wave vector $\boldsymbol{k}$.

(5.1.1) must satisfy the wave equation (4.2.8). Substituting (5.1.1) into (4.2.8) gives:

$$\begin{aligned} &\Delta\boldsymbol{E} - \mu_0\varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \\ &= -\boldsymbol{e}^{(1)}(k_x^2 + k_y^2 + k_z^2 - \mu_0\varepsilon_0\omega^2)\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t) \qquad (5.1.5) \\ &= 0 \end{aligned}$$

and

$$\omega^2\mu_0\varepsilon_0 = k_x^2 + k_y^2 + k_z^2 = k^2 \qquad (5.1.6)$$

have to be satisfied. Here the scalar wave number k is defined in (4.4.6).
Similarly, the magnetic field can be given as:

$$\boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{e}^{(2)} H_0 \cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t) \qquad (5.1.7)$$

Substituting (5.1.1) into the Gauss's law (4.2.3) we have:

$$\begin{aligned} div\boldsymbol{E}(\boldsymbol{r},t) &= \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) \\ &= -(e_x^{(1)}k_x + e_y^{(1)}k_y + e_z^{(1)}k_z)E_0 \sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t) \\ &= -(\boldsymbol{e}^{(1)}\cdot\boldsymbol{k})\sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t) = 0 \end{aligned} \qquad (5.1.8)$$

and similarly we have

$$div\boldsymbol{B}(\boldsymbol{r},t) = -(\boldsymbol{e}^{(2)}\cdot\boldsymbol{k})\sin(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t) = 0 \qquad (5.1.9)$$

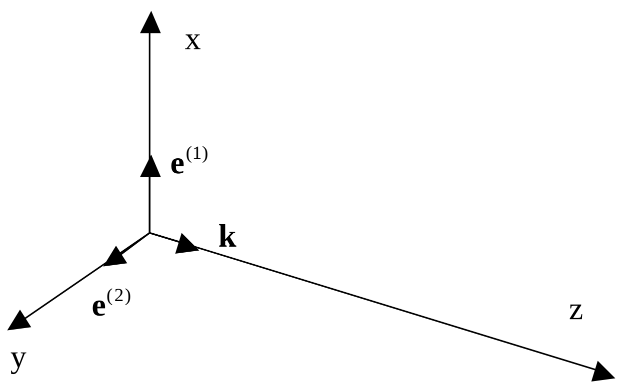
Then we obtain the relation

$$\boldsymbol{e}^{(1)}\cdot\boldsymbol{k} = \boldsymbol{e}^{(2)}\cdot\boldsymbol{k} = 0 \qquad (5.1.10)$$

(5.1.10) shows that the both electric and magnetic field is perpendicular to the wave vector $\boldsymbol{k}$. And substituting (5.1.1) and (5.1.7) into (4.2.1) we have:

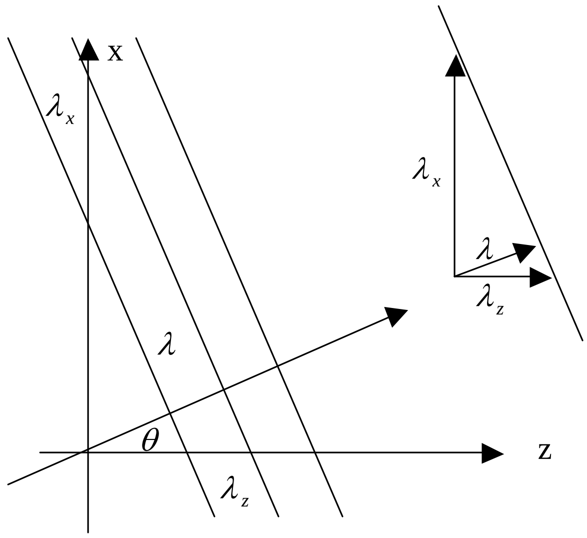
$$H_0\boldsymbol{e}^{(2)} = \boldsymbol{k}\times\boldsymbol{e}^{(1)}\frac{E_0}{\eta_0} \qquad (5.1.11)$$

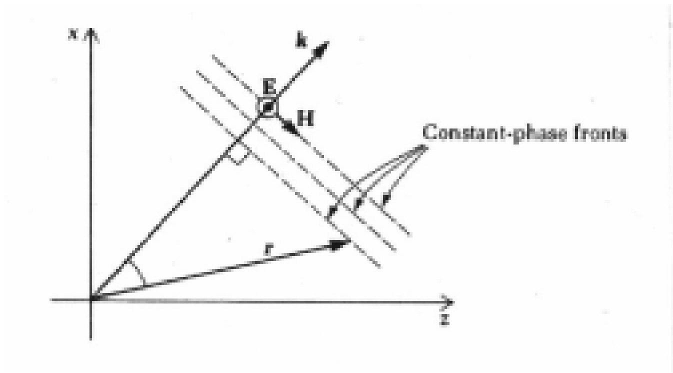
where η_0 is the intrinsic impedance of free space defined in (4.4.4). Now we can understand that the electric field, magnetic field and the wave vector of the plane wave is perpendicular to each other.



5.2 Two dimensional plane wave

Now we consider a plane wave in the two dimensional space. We define the wave vector be involved in the *x*-*y* plane and to the direction of θ measured from the z-axis.

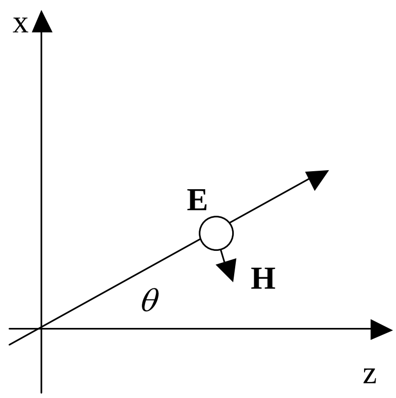




The wavelength λ is defined by the material, and is give as

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{\omega\sqrt{\mu_0\varepsilon_0}} \tag{5.2.1}$$

As shown in the figure, we can define apparent wave length along the x and z directions, which are λ_x and λ_z, respectively, and they are related to the apparent wave numbers as follows:



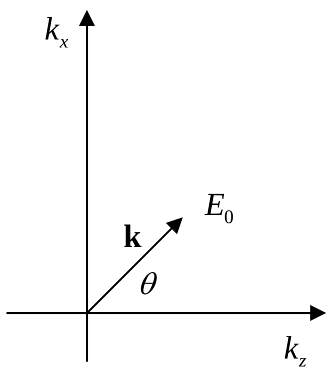
$$\begin{aligned} k_x &= \frac{2\pi}{\lambda_x} = \frac{2\pi}{\lambda}\sin\theta = k\sin\theta \\ k_z &= \frac{2\pi}{\lambda_z} = \frac{2\pi}{\lambda}\cos\theta = k\cos\theta \end{aligned} \tag{5.2.2}$$

The plane wave having the electric field to the *y*-axis direction can be given as:

$$\boldsymbol{E} = \hat{\boldsymbol{y}}E_0 e^{-j\boldsymbol{k}\cdot\boldsymbol{r}} = \hat{\boldsymbol{y}}E_0 e^{-jk_x x - jk_z z} \tag{5.2.3}$$

$$\boldsymbol{H} = -\frac{1}{j\omega\mu_0} rot\boldsymbol{E} = (-\hat{\boldsymbol{x}}k_z + \hat{\boldsymbol{z}}k_x)\frac{E_0}{\omega\mu_0} e^{-j\boldsymbol{k}\cdot\boldsymbol{r}} \tag{5.2.4}$$

$$\boldsymbol{k} = \hat{\boldsymbol{x}}k_x + \hat{\boldsymbol{z}}k_z \tag{5.2.5}$$



and

$$k = \omega\sqrt{\mu_0\varepsilon_0} \tag{5.2.6}$$

$$k_x = k\sin\theta \tag{5.2.7}$$

$$k_z = k\cos\theta \qquad (5.2.8)$$

It should be noted here that the plane wave propagating to an arbitrary direction, given by (5.2.3) and (5.2.4) is determined by only two terms, namely the wave vector $\boldsymbol{k}$ and the corresponding complex amplitude E_0.

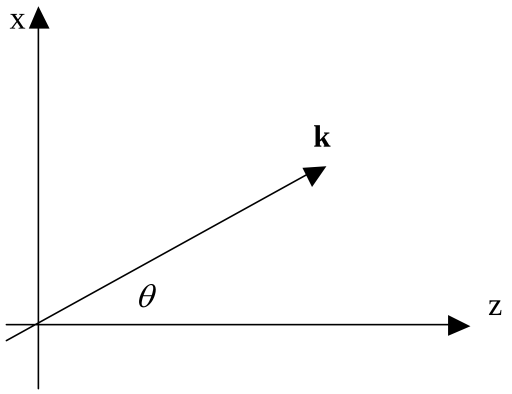
This relation is similar to the 2-D Fourier transformation:

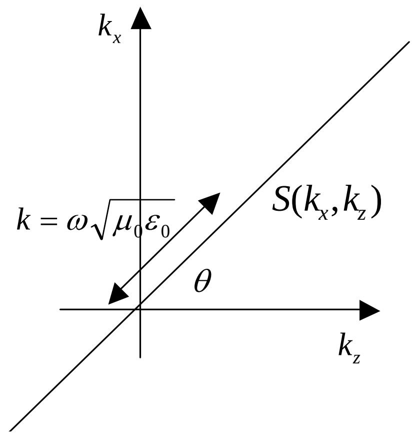
$$s(x,z) = \frac{1}{(2\pi)^2}\iint S(k_x,k_z)e^{-jk_x x - jk_z z}dk_x dk_z \qquad (5.2.9)$$

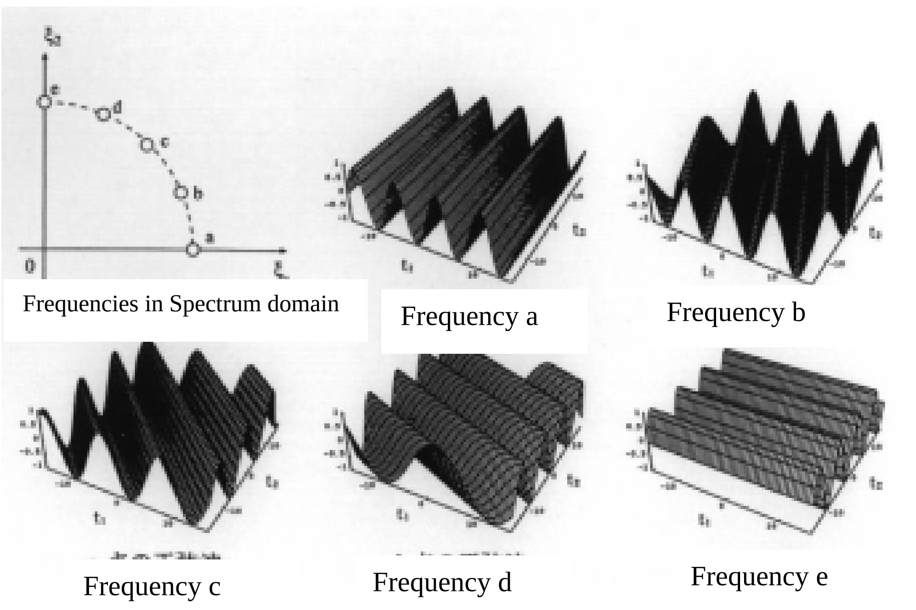
$$S(k_x,k_z) = \iint s(x,z)e^{jk_x x jk_z z}dxdz \qquad (5.2.10)$$

where the spectrum $S(k_x,k_z)$ corresponds to the complex amplitude of a plane wave propagating to the direction of the wave vector $\boldsymbol{k}$. And the waves, which propagates to the same direction, but different frequencies allies along a line of the direction of the wave vector.

Any wave propagating to any directions can be expressed as a summation of plane waves by using (5.2.9) and (5.2.10). This is called plane wave expansion of a wave.







Angular frequency (Ω_1,Ω_2) and corresponding $\cos(\Omega_1 t_1 + \Omega_2 t_2)$